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Pumping a Swing by Standing and Squatting

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Research in biomechanics supports the idea that humans perform skilled tasks as self-optimizing machines [1]–[4]. Accordingly, human motion can be viewed as a control problem entailing the minimization of a cost function with constraints on the control inputs and state variables. These constraints include magnitude bounds on the control inputs corresponding to the maximum forces and moments that the muscles can apply, as well as bounds on the state variables corresponding to the maximum displacements and rotations permitted by the joints and muscles. While these constraints can be determined by experimental and analytical means, researchers often disagree over the cost function [5].

Human locomotion, in particular walking and running, has been widely studied. The walking frequency is predicted in [6] by assuming a driven harmonic oscillator model for the lower body while minimizing the energy required to drive the oscillator. Other forms of locomotion such as rowing, swimming, and jumping have also been studied. For example, [7] deals with the optimization of muscle coordination for maximal-height jumping.

Models of the human body, such as those used in [6] and [7] to study walking and jumping, are typically based on a collection of appendages and joints. The complexity of these models often precludes analytical solutions to optimal control problems, thus necessitating numerical methods. In some cases, however, low-order models are sufficient. For example, a rider pumping a playground swing can be modeled adequately as a second-order system.

Swing Pumping

Pumping a swing is a classical problem that has been widely studied. Various pumping strategies such as standing [8] and sitting [9] have been considered. The rider has

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been modeled as a point mass [10], as a rigid body with a moment of inertia [11], and as an assembly of linked dumbbells [12]. In this article, we focus our attention on the problem of time-optimal pumping of a swing, where the rider pumps the swing by alternately standing and squatting.

Figure 1, which is a composite image created from photos taken at equal time intervals, depicts pumping by standing and squatting. As the figure shows, the child is squatting as she approaches the midpoint of the swing's motion, and she stands up near the midpoint. The child then travels toward the highest point while standing, and returns to the squatting position in the vicinity of the highest point. On the return journey, the child is again squatting while approaching the midpoint, and stands up near the midpoint, thus repeating the process. A look around the neighborhood playground reveals that children playing on swings generally follow the same strategy. In this article, we show that if the transitions to standing and squatting occur instantaneously, then this pumping strategy is time optimal (Figure 2).

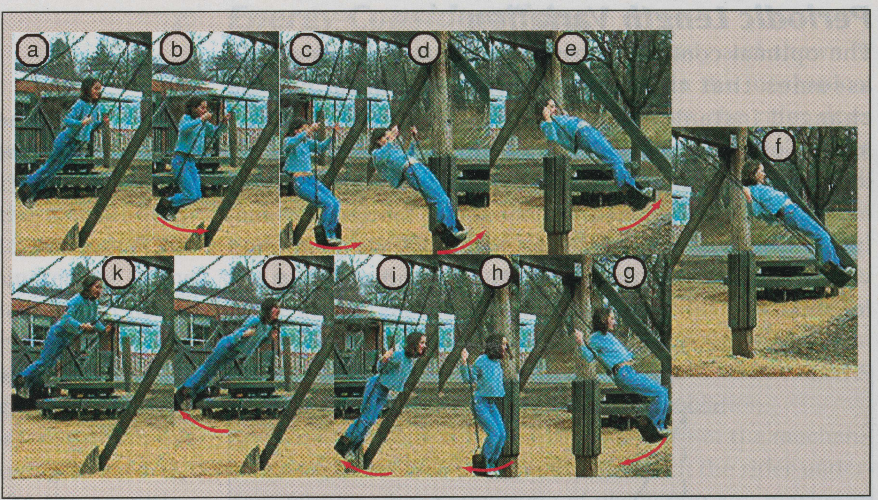

Figure 1. *An entire oscillation of a child pumping a swing. (a) The child at the start of the oscillation, (b) squatting and moving toward the midpoint of the trajectory, (c) beginning to stand up as she reaches the midpoint, and (d) and (e) standing up and moving toward the highest point. (f) She is at the highest point, and (g) on the return journey she is in a squatting position. (h) She is beginning to stand up, and (i) she is standing. (k) She reaches the highest point. The change in the amplitude of oscillation can be seen by comparing (a) and (k). The frames were taken 4/15th of a second apart. (Photos courtesy of Prof. Andy Ruina.)*

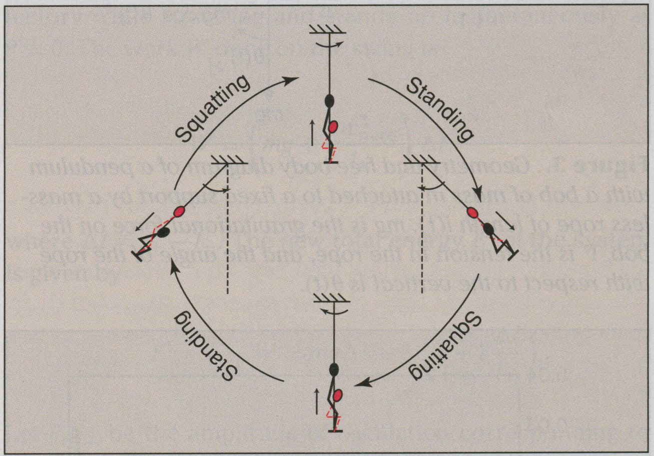

Figure 2. *Pumping strategy with instantaneous transitions from standing to squatting and vice versa. The 6 o'clock figure shows the rider at the lowest point of the swing's trajectory transitioning from a squatting position (dotted red) to a standing position (solid black). (Figure modified from [4].)*

Equations of Motion

The rider-and-swing system is modeled as a pendulum with a bob of mass m attached to a fixed support by a rope of variable length $l(t)$. The rope is taken to be massless, and the angle of the rope with respect to the vertical is denoted by $\theta(t)$ (Figure 3). Let l_+ and l_-, satisfying $0 < l_- < l_+$, denote the maximum and minimum lengths of the pendulum corresponding to squatting and standing, respectively. Let $L \triangleq (1/2)(l_+ + l_-)$ be the mean length of the pendulum. Dissipative forces such as bearing friction and wind drag are ignored. The length of the pendulum is the control input for the system.

Conservation of angular momentum for the pendulum gives

$$\frac{dH}{dt} = \tau,$$

where $H \triangleq ml^2\dot{\theta}$ is the angular momentum of the pendulum about the fixed support, and τ is the net torque about the fixed support due to all of the forces acting on the pendulum bob. The torque about the fixed support due to the tension Γ in the rope is zero, while the torque due to the gravitational force is $-mgl(t)\sin\theta(t)$. Therefore,

$$\frac{d}{dt}(l^2(t)\dot{\theta}(t)) = -gl(t)\sin\theta(t). \quad (1)$$

Hence,

$$\ddot{\theta} + \frac{2\dot{l}\dot{\theta}}{l} + \frac{g\sin\theta}{l} = 0, \quad (2)$$

where the dependence of l and θ on t has been dropped.

Periodic Length Variation

The optimal control problem considered in this article assumes that the length of the pendulum can be changed instantaneously. In practice, however, the motion of the rider results in a continuous change in the length of the pendulum, and as Figure 1 suggests, the motion can be idealized by sinusoidal motion. To analyze swing pumping when the length of the pendulum varies sinusoidally, define $\nu \triangleq l\theta$ [13] and use $\sin\theta \approx \theta$ for small θ in (2) to obtain

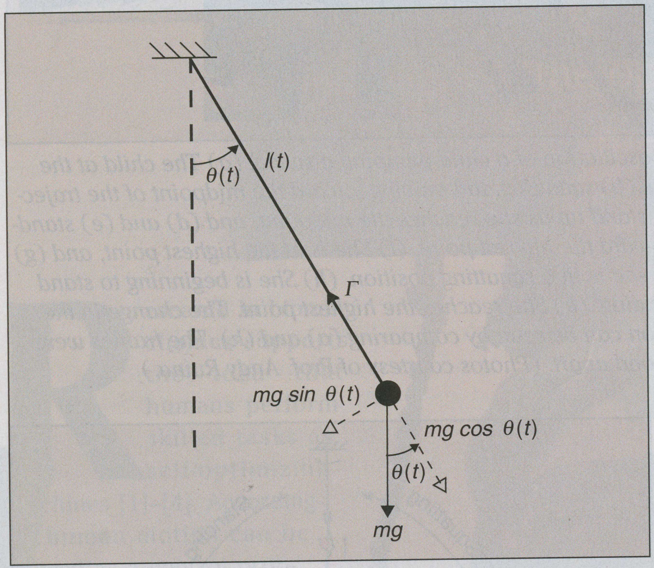

Figure 3. *Geometry and free-body diagram of a pendulum with a bob of mass m attached to a fixed support by a massless rope of length l(t). mg is the gravitational force on the bob, Γ is the tension in the rope, and the angle of the rope with respect to the vertical is θ(t).*

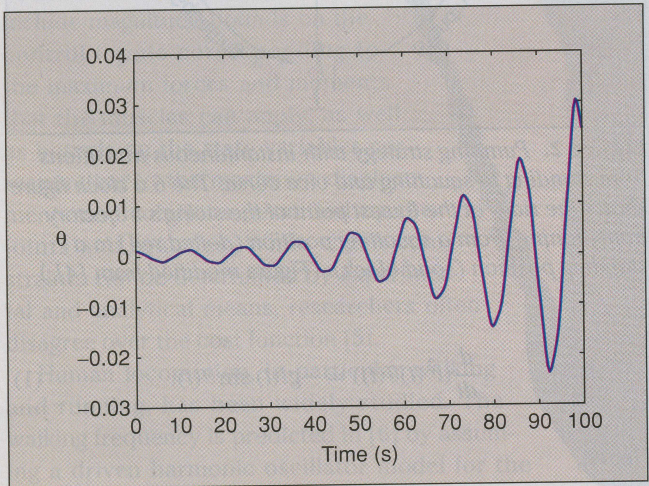

Figure 4. *Solution of Mathieu's equation for $\delta = 1/4$, $\epsilon = 0.1$, and $L = 10$. The solution to Mathieu's equation is unbounded for certain values of the parameters. The figure depicts the case in which the frequency of variation of the pendulum's length is twice the pendulum's natural frequency.*

$$\ddot{\nu} + \frac{1}{l}(g - \ddot{l})\nu = 0.$$

Rescaling time by substituting $\bar{t} \triangleq \omega t$ and letting the length of the pendulum vary as $l(t) = L(1 + \epsilon\cos\omega t)$, where $\epsilon = (l_+ - l_-/l_+ + l_-) < 1$, yields

$$\nu'' + \frac{\delta + \epsilon\cos\bar{t}}{1 + \epsilon\cos\bar{t}}\nu = 0,$$

where $()'$ denotes $d()/d\bar{t}$ and $\delta \triangleq g/(L\omega^2)$. As a first-order approximation in ϵ we obtain Mathieu's equation

$$\nu'' + (\delta + \epsilon(1 - \delta)\cos\bar{t})\nu = 0.$$

Mathieu's equation appears in different mathematical and engineering contexts, and its properties have been studied in detail [14]. For certain values of δ and ϵ, the solution to this equation is unbounded. One such solution for $\delta = 1/4$ and $\epsilon = 0.1$ is shown in Figure 4. For $\delta = 1/4$, the frequency of variation of the length of the pendulum is twice the pendulum's natural frequency $\sqrt{g/L}$. In fact, the rider in Figure 1 also completes two oscillations of the standing and squatting motion for every oscillation of the swing.

Instantaneous Variation of the Length

While thus far the length $l(t)$ of the pendulum has been taken to be a continuous function of time, optimal control theory allows $l(t)$ to be piecewise continuous. Following the discussion in [15], we use physical reasoning and elementary calculus to build intuition about swing pumping when $l(t)$ varies instantaneously.

Assume the rider in Figure 1 is initially squatting and moving toward the midpoint of the trajectory, corresponding to $\theta = 0$. Let t_0 denote the time corresponding to Figure 1(b) at which the rider is squatting and approaching the midpoint, and let $t_0 + \Delta t$ denote the later time instant corresponding to Figure 1(d) at which the rider is standing and traveling away from the midpoint. Let ε be such that $|\theta(t)| \le \varepsilon$ for $t \in [t_0, t_0 + \Delta t]$. Using the fact that the change in angular momentum is obtained by integrating the torque with respect to time, we have

$$l_-^2\dot{\theta}_{\rm stand} - l_+^2\dot{\theta}_{\rm squat} = -\int_{t_0}^{t_0+\Delta t} gl(t)\sin\theta(t)\,dt, \qquad (3)$$

where $\dot{\theta}_{\rm squat}$ and $\dot{\theta}_{\rm stand}$ are the angular velocities before and after standing, respectively. Since $|\sin\theta(t)| \le \varepsilon$ for $t \in [t_0, t_0 + \Delta t]$, the right-hand side of (3) is $O(\varepsilon)$, and as $\varepsilon \to 0$ we have

$$\theta_{\text{stand}} = \left(\frac{l_+}{l_-}\right)^2 \theta_{\text{squat}}. \quad (4)$$

Since $l_+ > l_-$, the angular velocity of the rider increases by this action and the rider attains a larger amplitude of oscillation. The rider thus increases the amplitude of oscillation by switching from squatting to standing at the midpoint of the motion. To impart this increment in the angular velocity repeatedly, the rider must return to the squatting position at another point along the motion, ideally without affecting the velocity or the angle.

Consider the point $\dot{\theta} \approx 0$ in the swing's motion, corresponding to the highest point in the rider's path. Suppose the rider is standing at time t_1 while moving toward the highest point [Figure 1(e)] and returns to the squatting position at time $t_1 + \Delta t$ while moving away from the highest point [Figure 1(g)]. Let $\varepsilon > 0$ satisfy $|\dot{\theta}(t)| \leq \varepsilon$ for $t \in [t_1, t_1 + \Delta t]$. Since the difference in the magnitude of $\dot{\theta}$ before and after squatting does not differ by more than 2ε, the angular velocity remains unaffected by this transition in the limit that the transition is carried out instantaneously at the highest point of the motion, that is, as $\varepsilon \to 0$. Furthermore, intuition suggests that if the transition is instantaneous, then the angle also remains unaffected. To verify this observation, we integrate (1) from t_1 to t, where $t \leq t_1 + \Delta t$, obtaining

$$l^2(t)\dot{\theta}(t) - l_-^2\dot{\theta}(t_1) = -\int_{t_1}^{t} gl(t)\sin\theta(t)\, dt. \quad (5)$$

Dividing by $l^2(t)$ and integrating from t_1 to $t_1 + \Delta t$ yields

$$\theta(t_1 + \Delta t) - \theta(t_1) - \int_{t_1}^{t_1+\Delta t}\left(\frac{l_-}{l(t)}\right)^2 \dot{\theta}(t_1)dt = \int_{t_1}^{t_1+\Delta t} \frac{-1}{l^2(t)} \int_{t_1}^{t} gl(\zeta)\sin\theta(\zeta)\, d\zeta\, dt. \quad (6)$$

In the limit of instantaneous transition so that the angular velocity is zero, that is, $\varepsilon \to 0$ and $\Delta t \to 0$ simultaneously, the integrals in (5) and (6) tend to zero. The angle is thus unaffected during the transition.

By standing and squatting at appropriate times during the motion, it follows from (4) that the rider increases the amplitude of oscillation of the swing. When the process is carried out in reverse, by standing at the highest point and squatting at the lowest point of the swing's trajectory, the amplitude of oscillation of the pendulum is decreased. The amplitude of oscillation can thus be increased or decreased by suitably altering the length of the pendulum.

Energy Considerations

The pumping strategy described previously and shown in Figure 2 leads to a geometric increase in the mechanical energy of the system [8]. In this strategy, the rider does the most work on the swing per oscillation, since the rider stands up at the lowest point, in the presence of the maximal gravitational and centrifugal forces, and squats at the highest point, where the centrifugal force vanishes and the component of the gravitational force along the length of the swing reaches its minimum. Since the work done on the swing is converted into stored energy, an increase in the mechanical energy of the system results in a corresponding increase in the amplitude of oscillation.

To demonstrate the geometric increase in the mechanical energy, consider the situation in which the rider undergoes motion whose amplitude of oscillation and maximum speed are given by $\theta_{\max}$ and $v_{\max}$, respectively. The total energy E of the system is thus given by

$$E = \frac{1}{2}mv_{\max}^2 = mgl_+(1 - \cos\theta_{\max}).$$

Suppose that the rider approaches the midpoint of the trajectory while squatting and stands up instantaneously at $\theta = 0$. The work W done on the swing is

$$W = \left(mg + \frac{mv_{\max}^2}{l_+}\right)\Delta l,$$

where $\Delta l \triangleq l_+ - l_-$. The new total energy E' of the system is given by

$$E' = E + W = mg\Delta l + E\left(1 + 2\frac{\Delta l}{l_+}\right).$$

Let $\theta_{\max}$ be the amplitude of oscillation corresponding to energy E'. The work W' done by the rider at the top of the swing's motion in returning to the squatting position is

$$W' = -mg\Delta l \cos\theta'_{\max}.$$

The total energy of the system E'' is then given by

$$E'' = E' + W' \approx E\left(1 + 3\frac{\Delta l}{l_+}\right),$$

where higher powers of $(\Delta l)/(l_+)$ have been ignored. Since this process is carried out twice per oscillation, the total energy E_n at the end of n oscillations is

$$E_n = E\left(1 + 3\frac{\Delta l}{l_+}\right)^{2n}$$

Thus, energy considerations show that the amplitude of oscillation of the swing can be increased by suitably changing the swing's length. Similarly, by interchanging the standing and squatting positions, the energy of the system can be reduced.

Optimal Control: Linearized Case

In this section, we analyze swing pumping as an optimal control problem. We begin by substituting $z_1 = \theta$ and $z_2 = \dot{\theta}$ in (2) to obtain

Human motion can be viewed as a control problem entailing the minimization of a cost function with constraints on the control inputs and state variables.

$$\dot{z}_1 = z_2,$$
$$\dot{z}_2 = -\frac{2\dot{l}z_2}{l} - \frac{g \sin z_1}{l}. \quad (7)$$

Note that the derivative $\dot{l}$ of the control input l appears in (7). Therefore, $\dot{l}$ is impulsive when l is discontinuous, and the state z_2 changes instantaneously. That is, as seen from (4), a discontinuous control input causes a discontinuity in the angular velocity. To remove the explicit appearance of the impulse, define $x_2 = z_2 l^2$ (the angular momentum) and $x_1 = z_1$ [16] to obtain the nonimpulsive system [17]

$$\dot{x}_1 = \frac{x_2}{l^2},$$
$$\dot{x}_2 = -gl \sin x_1. \quad (8)$$

Substituting $l(t) = L(1 + \epsilon u(t))$ in (8), where $\epsilon = (l_+ - l_-)/(l_+ + l_-) < 1$ and $|u(t)| \leq 1$, the first-order approximation in ϵ is

$$\dot{x}_1 = \frac{x_2}{L^2} - \frac{2\epsilon u x_2}{L^2},$$
$$\dot{x}_2 = -gL \sin x_1 - gL\epsilon u \sin x_1. \quad (9)$$

The squatting position with length l_+ corresponds to $u = +1$, and the standing position with length l_- corresponds to $u = -1$. Linearizing about the origin and setting $L = 1$ and $g = 1$ without any loss of generality, we obtain

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - u\begin{bmatrix} 0 & 2\epsilon \\ \epsilon & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \quad (10)$$

Rewriting in matrix form, we have the bilinear system [18], [19]

$$\dot{x} = Ax + uBx. \quad (11)$$

The controllability criterion for bilinear systems given in [19] shows that it is not possible to drive (11) from every initial state to every prescribed terminal state in finite time. For example, the state cannot be moved away from the origin, since the origin is a fixed point of the system. The physical implication is that a push is needed to start swinging. However, a push is not admissible in our problem definition. Also, it can be shown [19] that the origin cannot be reached in finite time by bounded controls. With these points in mind, we thus consider the problem of bringing the system (11) from an initial nonzero angle and zero angular velocity $x_{\rm ini} = [\bar{x}_1, 0]^{\rm T}$ to a target circle centered about the origin in minimum time. The control input is bounded in magnitude such that $|u(t)| \leq 1$.

The Pontryagin Minimum Principle

Having obtained the equations of motion and formulated the time-optimal control problem, we now derive the solution using optimal control theory [20]–[30]. Let $x^*(t)$ be the trajectory of (11) corresponding to the time-optimal control $u^*(t)$. Let ρ be the radius of the target circle centered about the origin. The optimal trajectory $x^*(t)$ satisfies the boundary conditions, $x^*(0) = [\bar{x}_1, 0]^{\rm T}$ and $x^*(T^*) \in S$, where $S \triangleq \{x : \|x\| = \rho\}$ and T^* is the optimal time to reach S. The Hamiltonian $\mathcal{H}$ is defined by

$$\mathcal{H}(x(t), p(t), u(t)) \triangleq 1 + p^{\rm T}(t)[Ax(t) + u(t)Bx(t)].$$

Then there exists a nontrivial costate vector $p^*(t)$ such that

- The differential equations

$$\dot{x}^*(t) = \frac{\partial \mathcal{H}}{\partial p}(x^*(t), p^*(t), u^*(t)), \quad (12)$$
$$\dot{p}^*(t) = -\frac{\partial \mathcal{H}}{\partial x}(x^*(t), p^*(t), u^*(t)), \quad (13)$$

  are satisfied by the state and the costate vectors with boundary conditions $x^*(0) = [\bar{x}_1, 0]^{\rm T}$ and $x^*(T^*) \in S$.

- The Hamiltonian is minimized by the optimal control input $u^*(t)$ for all $t \in [0, T^*]$, that is,

$$\mathcal{H}(x^*(t), p^*(t), u^*(t)) = \min_{|u(t)|\le 1} \mathcal{H}(x^*(t), p^*(t), u(t)). \quad (14)$$

- The Hamiltonian for the optimal control input is zero, that is, for all $t \in [0, T^*]$,

$$\mathcal{H}(x^*(t), p^*(t), u^*(t)) = 0. \quad (15)$$

- The costate vector $p^*(T^*)$ is transverse to S, that is, for $x \in M[x^*(T^*)]$,

$$p^{*\mathrm{T}}(T^*)[x - x^*(T^*)] = 0, \quad (16)$$

where $M[x^*(T^*)]$ is the tangent plane of S at $x^*(T^*)$, as shown in Figure 5.

Bang-Bang Control

Using (14), the optimal control is given by

$$u^*(t) = \mathrm{sgn}[2p_1^*(t)x_2^*(t) + p_2^*(t)x_1^*(t)]. \quad (17)$$

Since the argument of the sgn function vanishes only at isolated points, the optimal control is bang-bang, that is, u is piecewise constant and assumes the values +1 and −1.

State and Costate Vectors

It follows from (12) that $x^*(t)$ satisfies

$$\begin{bmatrix} \dot{x}_1^* \\ \dot{x}_2^* \end{bmatrix} = \begin{bmatrix} 0 & 1 - 2\epsilon u^* \\ -(1+\epsilon u^*) & 0 \end{bmatrix} \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix}, \quad (18)$$

with the initial condition $x^*(0) = [\bar{x}_1, 0]^{\mathrm{T}}$, where u^* is a piecewise constant function taking the values of +1 and −1. Rewriting (18) as

$$\ddot{x}_1^*(t) + \omega^2 x_1^*(t) = 0,$$

where $\omega \triangleq \sqrt{1 - \epsilon u^* - 2\epsilon^2 u^{*2}}$, we obtain

$$x_1^*(t) = \bar{x}_1 \cos \omega t,$$

$$x_2^*(t) = -\frac{\omega \bar{x}_1}{1 - 2\epsilon u^*} \sin \omega t. \quad (19)$$

Let us now turn our attention to the costate vector. From (13) we have

$$\begin{bmatrix} \dot{p}_1^* \\ \dot{p}_2^* \end{bmatrix} = \begin{bmatrix} 0 & 1 + \epsilon u^* \\ -(1 - 2\epsilon u^*) & 0 \end{bmatrix} \begin{bmatrix} p_1^* \\ p_2^* \end{bmatrix} \quad (20)$$

and (15) implies

$$1 + x_2^*(t)p_1^*(t)(1 - 2\epsilon u^*) - x_1^*(t)p_2^*(t)(1 + \epsilon u^*) = 0. \quad (21)$$

It follows from (20) and (21) that

$$p_1^*(t) = \frac{1}{\bar{x}_1 \omega} \cos(\omega t + \phi),$$

$$p_2^*(t) = -\frac{1}{\bar{x}_1(1 + \epsilon u^*)} \sin(\omega t + \phi), \quad (22)$$

where ϕ is a parameter that is chosen based on the initial and final conditions of the optimal control problem.

Solving the Time-Optimal Control Problem

We now determine ϕ in (22) so that equations for the state and costate vectors satisfy all of the necessary conditions for optimality. Since the target set S is a circle, it follows from the transversality condition (16) that at the final time T^*, the state and costate vector are parallel to each other (Figure 5). Therefore,

$$p^*(T^*) = \lambda x^*(T^*),$$

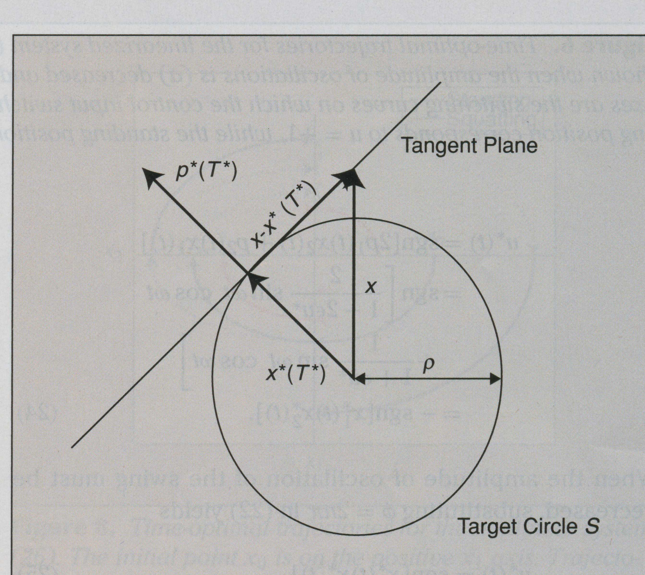

Figure 5. *Transversality condition. At the terminal time T^* the costate vector $p^*(T^*)$ is transverse to the tangent plane of S at $x^*(T^*)$.*

and thus

$$\frac{p_1^*(T^*)}{x_1^*(T^*)} = \frac{p_2^*(T^*)}{x_2^*(T^*)} \tag{23}$$

when $x_1^*(T^*)$ and $x_2^*(T^*)$ are nonzero.

Substituting (19) and (22) in (23) and using $\epsilon < 1$, we obtain $\sin\phi = 0$. It can be shown that $\sin\phi = 0$ when either $x_1^*(T^*)$ or $x_2^*(T^*)$ is zero. Therefore, either $\phi = 2n\pi$ or $\phi = (2n+1)\pi$, where $n = 0, 1, 2 \ldots$. When the swing must be pumped to increase its amplitude of motion, substituting $\phi = (2n+1)\pi$ in (22) and using (17) and (19) we have

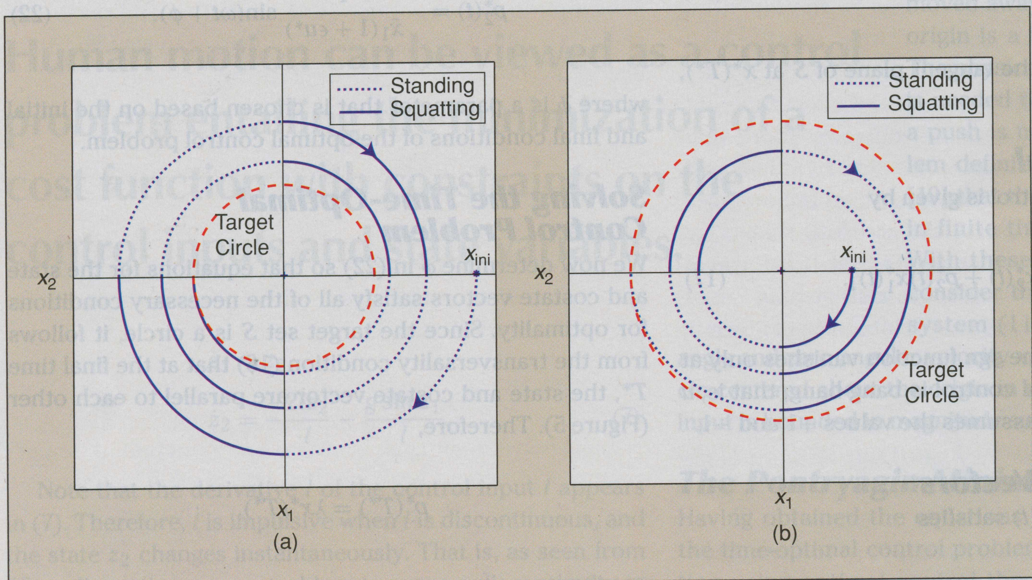

Figure 6. *Time-optimal trajectories for the linearized system (10). Time-optimal trajectories are shown when the amplitude of oscillations is (a) decreased and (b) increased. The coordinate axes are the switching curves on which the control input switches between +1 and −1. The squatting position corresponds to u = +1, while the standing position corresponds to u = −1.*

$$\begin{aligned} u^*(t) &= \text{sgn}[2p_1(t)x_2(t) + p_2(t)x_1(t)] \\ &= \text{sgn}\left[\frac{2}{1-2\epsilon u^*}\sin\omega t\ \cos\omega t\right. \\ &\quad \left. + \frac{1}{1+\epsilon u^*}\sin\omega t\ \cos\omega t\right] \\ &= -\,\text{sgn}[x_1^*(t)x_2^*(t)]. \end{aligned} \tag{24}$$

When the amplitude of oscillation of the swing must be decreased, substituting $\phi = 2n\pi$ in (22) yields

$$u^*(t) = \text{sgn}[x_1^*(t)x_2^*(t)]. \tag{25}$$

Optimal trajectories for the two cases are shown in Figure 6. The optimal control strategies (24) and (25) are in feedback form, and (24) corresponds to the pumping strategy shown in Figure 2, where the rider stands up at the lowest point and squats at the highest point.

Optimal Control: Nonlinear Case

To solve the time-optimal control problem for the nonlinear system (8), we use techniques from geometric optimal control [24]. Replacing l by u as the control input in (8), we have

$$\dot{x} = h(x,u) \triangleq \begin{pmatrix} \dfrac{x_2}{u^2} \\ -gu\sin x_1 \end{pmatrix}, \tag{26}$$

where $x \triangleq [x_1, x_2]^{\mathrm{T}}$ and with $l_- \le u(t) \le l_+$.

The main difficulty in analyzing (26) is the nonlinear dependence on the control input. However, defining $h^{\pm}(x) = h(x, l_{\pm})$, we observe that the set of admissible velocities at a point x is contained in the convex hull of the vectors 0, $h^-(x)$, and $h^+(x)$, as shown in Figure 7. Therefore, every admissible velocity at x for (26) is realized with a greater magnitude by a point on the segment joining $h^-(x)$ and $h^+(x)$, implying that points on this segment result in faster trajectories [25]. This observation enables us to solve the optimal control problem for the nonlinear system (26) by considering an auxiliary system with velocities in the segment joining $h^-(x)$ and $h^+(x)$. To illustrate this idea, we introduce the auxiliary system

$$\dot{x} = F(x) + G(x)v, \quad |v| \le 1, \tag{27}$$

where

$$F(x) \triangleq \begin{pmatrix} \dfrac{c}{2a}x_2 \\ -\dfrac{g}{2}b\sin x_1 \end{pmatrix}, \quad G(x) \triangleq \begin{pmatrix} \dfrac{b\,\Delta l}{2a}x_2 \\ -\dfrac{g}{2}\Delta l\sin x_1 \end{pmatrix},$$

and $a \triangleq (l_+ l_-)^2$, $b \triangleq l_+ + l_-$, and $c \triangleq l_+^2 + l_-^2$. The auxiliary system corresponds exactly to having velocities in the segment joining $h^-(x)$ with $h^+(x)$, which is easily checked by verifying that $F(x) + G(x) = h^+(x)$ and $F(x) - G(x) = h^-(x)$.

Dimension-two systems of type (27) have been studied in [24] and [26]–[28]. For these systems, a detailed analysis of the structure of optimal trajectories and a synthesis of the optimal control is possible. A general method for synthesizing an optimal control on a two-dimensional (2-D) manifold is illustrated in [26], along with a classification of various singularities that appear in optimal flows.

A key role in the analysis of optimal trajectories for (27) is played by the functions

$$\begin{aligned}\Delta_A(x) &\triangleq \det(F(x), G(x)) \\ &= F_1(x)G_2(x) - F_2(x)G_1(x), \\ \Delta_B(x) &\triangleq \det(G(x), [F, G](x)) \\ &= G_1(x)[F, G]_2(x) - G_2(x)[F, G]_1(x),\end{aligned}$$

where the Lie bracket $[F, G]$ of F and G is given by $[F, G] \triangleq \nabla G \cdot F - \nabla F \cdot G$. In particular, if neither function vanishes in an open set Ω of the phase space, then every optimal trajectory in Ω is bang-bang with, at most, one switching [24]. For (27) we compute

$$\begin{aligned}\Delta_A(x) &= -\frac{g\Delta l(c + b^2)}{4a} x_2 \sin x_1, \\ \Delta_B(x) &= \frac{g\Delta l^2(c + b^2)}{8a}\left(\frac{b}{a} x_2^2 \cos x_1 + g \sin^2 x_1\right),\end{aligned}$$

implying only bang-bang controls for $x_1 \in [-\pi/2, \pi/2]$ and, at most, one switching for every quadrant.

The link between (26) and (27) is given by the following theorem [25].

Theorem 1

Consider two points x_0 and x_f, denoting the prescribed initial and final states of the nonlinear system (26).

- If there exists a bang-bang time-optimal control v steering (27) from x_0 to x_f along trajectory γ, then there exists a time-optimal control u for (26) corresponding to the same trajectory γ of v, that is, $h(\gamma(t), u(t)) = F(\gamma(t)) + G(\gamma(t))v(t)$.
- If the time-optimal control v for (27) is not bang-bang, then a time-optimal control for (26) does not exist. In this case, let T denote the time taken by v. Then for each $\varepsilon > 0$ there exists a control u steering x_0 to x_f in time $T + \varepsilon$, such that $u(t) \in \{l_\pm\}$ for every t.

Roughly speaking, Theorem 1 states that either a bang-bang optimal control exists for (27) and that the same trajectories are optimal for (26), or the optimal control for (27) is not bang-bang, in which case the optimal control for (27) does not exist, but the minimum-time trajectory of (26) is approximated by bang-bang trajectories. Therefore, to obtain optimal controls for (26), it suffices to construct time-optimal controls for (27).

Figure 8, which is constructed using the methods developed in [26], shows time-optimal trajectories for the nonlinear system (26) starting from x_0 on the positive $x_1$axis. The dashed green lines indicate the switching curves for which the control input changes sign along the optimal trajectories. The trajectories that minimize and maximize the final oscillation correspond to bang-bang controls, with the control input switching each time the state crosses the coordinate

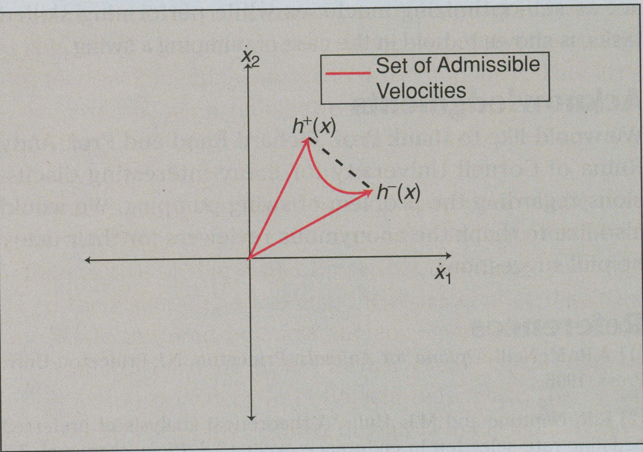

Figure 7. *The set of admissible velocities for the nonlinear system (26). The set of admissible velocities for the state x is a proper subset of the convex hull of the vectors 0, $h^-(x)$, and $h^+(x)$. The figure is constructed for the case $x_1 < 0$ and $x_2 > 0$.*

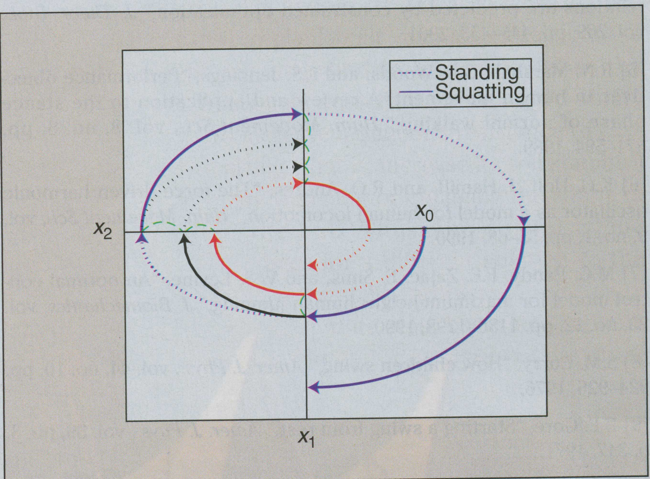

Figure 8. *Time-optimal trajectories for the nonlinear system (26). The initial point x_0 is on the positive x_1 axis. Trajectories minimizing (red) and maximizing (blue) the amplitude of oscillation are obtained when the control input switches between +1 and −1 as the state crosses the coordinate axes. The dashed green lines are the switching curves.*

axes. The optimal control for the nonlinear system is identical to the solution obtained for the linearized system.

Conclusions

Children playing on a swing generally stand up at the lowest point and squat at the highest. Under the assumption that standing and squatting can be carried out instantaneously, we have shown, using the Pontryagin minimum principle and techniques from geometric optimal control, that this pumping strategy is time optimal for the problem of maximizing the amplitude of oscillation of a swing. Given the exuberant nature of youth, it is reasonable to believe that children on a swing try to go as high as possible as quickly as possible. Thus, one of the key concepts in biomechanics, that humans act as self-optimizing machines while performing skilled tasks, is shown to hold in the case of pumping a swing.

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References

[1] A.R. McNeill, *Optima for Animals*. Princeton, NJ: Princeton Univ. Press, 1996.

[2] R.R. Neptune and M.L. Hull, "A theoretical analysis of preferred pedaling rate selection in endurance cycling," *J. Biomechan.*, vol. 33, no. 4, pp. 409–415, 2000.

[3] E. Todorov and W. Li, "Optimal control methods suitable for biomechanical systems," in *Proc. 25th Annu. Int. Conf. IEEE Eng. Biology Medicine Soc.*, Cancun, Mexico, 2003, vol. 2, pp. 1758–1761.

[4] J.E.A. Bertram and A. Ruina, "Multiple walking speed-frequency relations are predicted by constrained optimization," *J. Theor. Biol.*, vol. 209, pp. 445–453, 2001.

[5] R.N. Marshall, G.A. Woods, and L.S. Jennings, "Performance objectives in human movement: A review and application to the stance phase of normal walking," *Hum. Movement Sci.*, vol. 8, no. 6, pp. 571–594, 1989.

[6] K.G. Holt, J. Hamill, and R.O. Andres, "The force driven harmonic oscillator as a model for human locomotion," *Hum. Movement Sci.*, vol. 9, no. 1, pp. 55–68, 1990.

[7] M.G. Pandy, F.E. Zajac, E. Sims, and W.S. Levine, "An optimal control model for maximum-height human jumping," *J. Biomechanics*, vol. 23, no. 12, pp. 1185–1198, 1990.

[8] S.M. Curry, "How children swing," *Amer. J. Phys.*, vol. 44, no. 10, pp. 924–926, 1976.

[9] B.F. Gore, "Starting a swing from rest," *Amer. J. Phys.*, vol. 39, no. 3, p. 347, 1971.

[10] P.L. Tea and H. Falk, "Pumping on a swing," *Amer. J. Phys.*, vol. 36, no. 12, pp. 1165–1166, 1968.

[11] W.B. Case, "The pumping of a swing from the standing position," *Amer. J. Phys.*, vol. 64, no. 3, pp. 215–220, 1996.

[12] W.B. Case and M.A. Swanson, "The pumping of a swing from the seated position," *Amer. J. Phys.*, vol. 58, no. 5, pp. 463–467, 1990.

[13] J.A. Burns, "More on pumping a swing," *Amer. J. Phys.*, vol. 38, no. 7, pp. 920–922, 1970.

[14] N.W. McLachlan, *Theory and Applications of Mathieu Functions*. New York: Dover, 1964.

[15] S. Wirkus, R. Rand, and A. Ruina, "How to pump a swing," *College Math. J.*, vol. 29, no. 4, pp. 266–275, 1998.

[16] A. Bressan, "On some control problems concerning the ski or swing," *Atti. Accad. Naz. Lincei, Memoirs*, vol. IX, no. 1, pp. 149–196, 1991.

[17] A. Bressan, "Impulsive control systems," in *Nonsmooth Analysis and Geometric Methods in Deterministic Optimal Control*, B.S. Mordukhovich and H.J. Sussmann, Eds. New York: Springer-Verlag, 1993, vol. 78, pp. 1–22.

[18] D.L. Elliott, "Bilinear systems," in *Wiley Encyclopedia of Electrical and Electronics Engineering*, J.G. Webster, Ed. New York: Wiley, 1999.

[19] R.R. Mohler, *Bilinear Control Processes*. New York: Academic, 1973.

[20] M. Athans and P.L. Falb, *Optimal Control*. New York: McGraw-Hill, 1966.

[21] A.E. Bryson and Y. Ho, *Applied Optimal Control*. Levittown, PA: Taylor and Francis, 1975.

[22] V. Jurdjevic, *Geometric Control*. Cambridge, U.K.: Cambridge Univ. Press, 1997.

[23] H. Hermes and J.P. LaSalle, *Functional Analysis and Time Optimal Control*. New York: Academic, 1969.

[24] H.J. Sussmann, "The structure of time-optimal trajectories for single-input systems in the plane: The C^∞ nonsingular case," *SIAM J. Control Optim.*, vol. 25, no. 2, pp. 433–465, 1987.

[25] B. Piccoli, "Time-optimal control problems for the swing and the ski," *Int. J. Control*, vol. 62, no. 6, pp. 1409–1429, 1995.

[26] U. Boscain and B. Piccoli, *Optimal Syntheses for Control Systems on 2-D Manifolds*. Berlin, Germany: Springer, 2004.

[27] A. Bressan and B. Piccoli, "A generic classification of time-optimal planar stabilizing feedbacks," *SIAM J. Control Optim.*, vol. 36, no. 1, pp. 12–32, 1998.

[28] B. Piccoli, "Classification of generic singularities for the planar time-optimal synthesis," *SIAM J. Control Optim.*, vol. 34, no. 6, pp. 1914–1946, 1996.

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